

Generating functions for intersection numbers on moduli spaces of curves

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Abstract

Using the connection between intersection theory on the Deligne-Mumford spaces $\overline{\mathcal{M}}_{g,n}$ and the edge scaling of the GUE matrix model (see [12, 14]), we express the n -point functions for the intersection numbers as n -dimensional error-function-type integrals and also give a derivation of Witten's KdV equations using the higher Fay identities of Adler, Shiota, and van Moerbeke.

1 Introduction

1.1 Overview

1.1.1

This paper is a continuation of [12] and is very closely related to [14]. It was observed in [12] and conceptually explained in [14] that the intersection theory on the moduli space of curves is closely connected to the edge scaling of the standard GUE matrix model (which can be called the edge-of-the-spectrum matrix model). Leaving the detailed discussion of this phenomenon to [14], we obtain here some formal consequences of this connection, most importantly, an error-function-type integral formula for certain generation functions for the intersection numbers known as the n -point functions. We also show how to derive, using results of Adler, Shiota, and van Moerbeke [1], the KdV equations from this matrix model.

1.1.2

Let $\overline{\mathcal{M}}_{g,n}$ be the stable compactification of the moduli space of genus g curves with n marked points. Let ψ_i be the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the i th point. We use the standard notation

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}, \quad \sum d_i = 3g - 3 + n, \quad (1.1)$$

for the intersection numbers of these ψ -classes.

The celebrated conjecture of Witten [17] says that for the following generating function

$$\mathbf{F}(t_0, t_1, \dots) = \sum_{(k_0, k_1, \dots)} \langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \prod \frac{t_i^{k_i}}{k_i!},$$

the exponential $\exp(\mathbf{F})$ is a τ -function for the KdV hierarchy in variables

$$T_{2i+1} = \frac{t_i}{(2i+1)!!}.$$

This conjecture was inspired by matrix models of 2-dimensional quantum gravity, see for example [4, 5] for a survey.

The proof of Witten's conjecture given by Kontsevich [11] uses another matrix model interpretation of the generating function \mathbf{F} .

1.1.3

The purpose of this paper is to exploit yet another random matrix connection [12] to evaluate a different generating function for (1.1), called the n -point function, in a closed form.

Definition 1.1. We call the following generating function

$$\mathcal{F}(x_1, \dots, x_n) = \sum_{g=0}^{\infty} \mathcal{F}_g(x_1, \dots, x_n),$$

where

$$\mathcal{F}_g(x_1, \dots, x_n) = \sum_{\sum d_i = 3g - 3 + n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \prod x_i^{d_i},$$

the n -point function.

The 2 and 3-point function were computed by Dijgraaf and Zagier, respectively. Several specialization of the 3-point function can be found in the paper [6]. Our main result, see Theorem 1 below, is that the n -point function \mathcal{F} is a certain specific multivariate error function (and hence, in particular, a multivariate hypergeometric function in the sense of Gelfand, Kapranov, and Zelevinsky, see Section 1.2.3).

1.1.4

We recall, referring the reader to [12, 14] for details, that the n -point functions turn out to be connected to the asymptotics of the following averages over $N \times N$ Hermitian matrices

$$\left\langle \prod_{k=1}^n \text{tr} H^{2[x_k N^{2/3}]} \right\rangle_N, \quad N \rightarrow \infty, \quad (1.2)$$

with respect to the standard Gaussian measure. Here $[\cdot]$ stands for the integer part. The asymptotics of (1.2) is equivalent to the knowledge of the distribution of eigenvalues of a random matrix near the edge of the Wigner semicircle. This distribution is well known to be described by the Airy ensemble, see for example [16]. We also note that this Airy behavior near the edge of the spectrum is very universal for random matrix ensembles, see for example [15].

1.1.5

By a standard application of the Wick formula, the asymptotics (1.2) is equivalent to the asymptotics in the following combinatorial enumeration problem. Consider a surface Σ and a map on it with n cells, that is, a way to glue Σ out of n polygons by identifying the edges of the polygons in pairs. The asymptotics of the number of ways this can be done, as n is fixed and the perimeters of the polygons go to infinity at certain relative rates, is encoded in the asymptotics of (1.2). A typical map with 3 cells on a genus $g = 2$ surface Σ is shown in Figure 1. Here one portion of the map is magnified so that to show the dendriform pattern that the map forms because the polygons are allowed to be glued to itself. In fact, the overwhelming part of the perimeter is typically contained in these trees as the perimeter goes to infinity. The macroscopic data of the map in Figure 1 is a trivalent graph

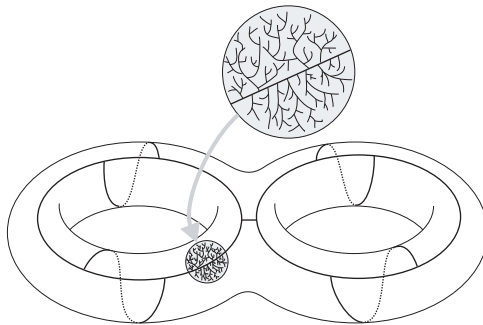


Figure 1: A typical map of large perimeter

embedded in Σ which is the same data as in [11], see [12, 14] for details of the connection of this approach to [11].

The asymptotics in this map enumeration problem is described by a certain function $\text{map}_\Sigma(x_1, \dots, x_n)$, see Section 2.1.7 of [12] for precise definitions and Section 2.1 below for a formula for map_Σ . When the surface Σ is connected of genus g , we write map_g instead of map_Σ . Obviously, the asymptotics map_Σ is multiplicative in connected components.

There exist a number of other combinatorial asymptotics equivalent to the asymptotics (1.2), such as the distribution of increasing subsequences in a random permutation, see for example [2, 13, 3, 8].

1.1.6

Another product of our analysis is an alternative derivation of Witten's KdV equations using the edge matrix model. It is based on the Fay identities methods developed by Adler, Shiota, and van Moerbeke in [1].

1.2 Formula for the n -point function

1.2.1 The function \mathcal{E}

The key ingredient in our formula for the n -point function will be the following function of x_1, \dots, x_n

$$\mathcal{E}(x_1, \dots, x_n) = \frac{1}{2^n \pi^{n/2}} \frac{\exp\left(\frac{1}{12} \sum x_i^3\right)}{\prod \sqrt{x_i}} \times \int_{s_i \geq 0} ds \exp\left(-\sum_{i=1}^n \frac{(s_i - s_{i+1})^2}{4x_i} - \sum_{i=1}^n \frac{s_i + s_{i+1}}{2} x_i\right), \quad (1.3)$$

where the integral is over $\mathbb{R}_{\geq 0}^n$ and $s_{n+1} = s_1$.

This integral admits a nice probabilistic interpretation, namely

$$\mathcal{E}(x) = \exp\left(\frac{1}{12} \sum x_i^3\right) \int_{\gamma \geq 0} e^{-\int \gamma} W(d\gamma),$$

where the integral is over the space of nonnegative piecewise linear function of the form shown in Figure 2 subject to the bridge condition $s_{n+1} = s_1$.

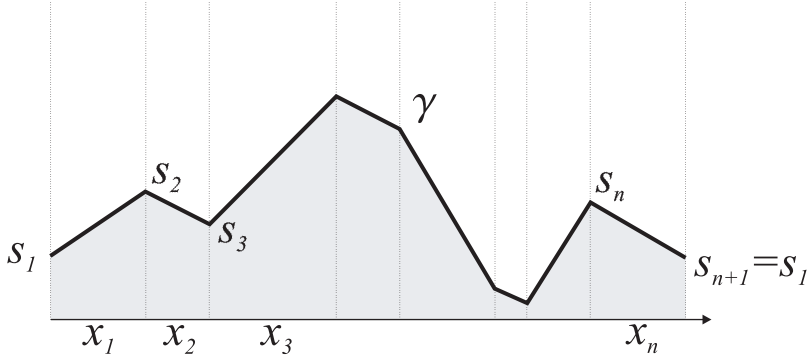


Figure 2: A piecewise linear function γ

The measure W is the natural Gaussian measure: the increments of γ are independent normal variables with variance equal to twice the length of the interval.

1.2.2 Main result

The function \mathcal{E} is clearly invariant under a cyclic shift of the x_i 's. We want to have something symmetric in the x_i 's, so we set

$$\mathcal{E}^\circ(x) = \sum_{\sigma \in S(n)/(12\dots n)} \mathcal{E}(x_{\sigma(1)}, \dots, x_{\sigma(s)}),$$

where the summation is over coset representatives modulo the cyclic group generated by the permutation $(12\dots n)$.

Let Π_n denote the set of all partitions α of the set $\{1, \dots, n\}$ into disjoint union of subsets. For any partition $\alpha \in \Pi_n$ with $\ell = \ell(\alpha)$ blocks, let x_α denote the vector of size ℓ formed by sums of x_i over the blocks of α . For example, if

$$\alpha = \{1, 3\} \sqcup \{2\} \in \Pi_3$$

then $\ell(\alpha) = 2$ and $x_\alpha = (x_1 + x_3, x_2)$.

By definition, we set

$$\mathcal{G}(x_1, \dots, x_n) = \sum_{\alpha \in \Pi_n} (-1)^{\ell(\alpha)+1} \mathcal{E}^\circ(x_\alpha).$$

For example,

$$\mathcal{G}(x_1, x_2) = \mathcal{E}(x_1 + x_2) - \mathcal{E}(x_1, x_2),$$

and, similarly,

$$\begin{aligned} \mathcal{G}(x_1, x_2, x_3) &= \mathcal{E}(x_1 + x_2 + x_3) - \mathcal{E}(x_1 + x_2, x_3) - \mathcal{E}(x_1 + x_3, x_2) \\ &\quad - \mathcal{E}(x_2 + x_3, x_1) + \mathcal{E}(x_1, x_2, x_3) + \mathcal{E}(x_1, x_3, x_2). \end{aligned}$$

In this notation, our result is

Theorem 1.

$$\mathcal{F}(x_1, \dots, x_n) = \frac{(2\pi)^{n/2}}{\prod x_i^{1/2}} \mathcal{G}\left(\frac{x}{2^{1/3}}\right). \quad (1.4)$$

We remark that the right-hand side of (1.4) makes sense for all $g \geq 0$ and $n \geq 1$, even for the pairs

$$(g, n) = (0, 1), (0, 2)$$

for which the corresponding moduli space is problematic. The corresponding terms, however, are not polynomial in x , see Section 2.6.3.

1.2.3

The integral (1.3) fits inside a more general class of integrals of the form

$$I(Q) = \int_{\mathbb{R}_{\geq 0}^n} e^{Q(s)} ds, \quad \deg Q = 2.$$

Such a multivariate analog of the error function is, as a function of coefficients of the polynomial Q , a multivariate hypergeometric function in the sense of Gelfand, Kapranov, and Zelevinsky. In fact, this is true for any polynomial

$$Q = \sum_{m=(m_1, \dots, m_n)} a_m s^m,$$

because the integral $I(Q)$ satisfies the obvious equations

$$\left(\prod \frac{\partial}{\partial a_{\mu^{(i)}}} - \prod \frac{\partial}{\partial a_{\nu^{(i)}}} \right) I(Q) = 0, \quad \sum \mu^{(i)} = \sum \nu^{(i)},$$

and integration by parts gives

$$I(Q) = - \int_{\mathbb{R}_{\geq 0}^n} e^{Q(s)} s_i \frac{\partial}{\partial s_i} Q ds,$$

which is equivalent to the homogeneity equations

$$\left(\sum_m m_i a_m \frac{\partial}{\partial a_m} + 1 \right) I(Q) = 0,$$

for $i = 1, \dots, n$.

2 Proof of the n -point function formula

2.1

We will assume that the reader is familiar with Section 2 of [12]. The main object we need from [12] is the function $\text{map}_g(x)$ which describes the $N \rightarrow \infty$ asymptotics of the number of genus g maps with n cells with perimeters $\sim Nx_1, \dots, \sim Nx_n$, respectively.

We have the following formula which is Theorem 3 in [12]

$$\int_{\mathbb{R}_{\geq 0}^n} e^{-(z,x)} \text{map}_g(x) \frac{dx}{x} = \frac{1}{2^{|e(\Gamma)|/2-1}} \sum_{\Gamma \in \Gamma_{g,n}^3} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{e \in e(\Gamma)} \frac{1}{\sqrt{z_{1,e}} + \sqrt{z_{2,e}}}, \quad (2.1)$$

where the summation is over all trivalent ribbon graphs Γ of genus g with n cells, the product is over all edges e of Γ , $z_{1,e}$ and $z_{2,e}$ are the z_i 's corresponding to the two sides of edge e , and

$$|e(\Gamma)| = 6g - 6 + 3n$$

is the number of edges of any graph $\Gamma \in \Gamma_{g,n}^3$.

On the other hand, we have the following formula of Kontsevich which is the unique boxed formula in [11]

$$\sum_{\Gamma \in \Gamma_{g,n}^3} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{e \in e(\Gamma)} \frac{1}{z_{1,e} + z_{2,e}} = 2^{-d-|e(\Gamma)|/3} \sum_{\sum d_i = 3g-3+n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle \prod_{i=1}^n \frac{(2d_i)!}{d_i!} z_i^{-2d_i-1}. \quad (2.2)$$

We refer the reader to [14] for a very detailed discussion and proof of this formula.

From (2.1) and (2.2), using the formula

$$\int_0^\infty e^{-st} t^{m-\frac{1}{2}} dt = \Gamma\left(m + \frac{1}{2}\right) s^{-m-\frac{1}{2}} = \sqrt{\pi} \frac{(2m)!}{2^{2m} m!} s^{-m-\frac{1}{2}}$$

and the fact that the function map_g is homogeneous of degree $3g - 3 + 3n/2$, we obtain the following:

Proposition 2.1.

$$\mathcal{F}_g(x_1, \dots, x_n) = \frac{\pi^{n/2}}{2^g} \frac{\text{map}_g(2x_1, \dots, 2x_n)}{\prod x_i^{1/2}}. \quad (2.3)$$

2.2

The function map_g can be obtained as the Laplace transform of a certain natural object. Introduce the Airy kernel, see for example [16],

$$\mathbf{K}(x, y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y},$$

where $\text{Ai}(x)$ is the classical Airy function.

The kernel \mathbf{K} differs by scaling of variables from the kernel K used in [12]. It is natural to use \mathbf{K} when working with $\text{map}_g(2x)$ instead of $\text{map}_g(x)$. We now introduce the corresponding modifications of the functions from Section 2.1.8 of [12].

Let the function \mathbf{R} be the following Laplace transform

$$\mathbf{R}(\xi_1, \dots, \xi_n) = \int_{\mathbb{R}^n} e^{(\xi, x)} \det [\mathbf{K}(x_i, x_j)] dx. \quad (2.4)$$

By definition, set

$$\mathbf{H}(x_1, \dots, x_n) = \sum_{\alpha \in \Pi_n} \mathbf{R}(x_\alpha), \quad (2.5)$$

where Π_n denotes the set of all partitions of the set $\{1, \dots, n\}$ into disjoint union of subsets, and x_α is the vector formed by sums of x_i over blocks of α . For example,

$$\begin{aligned} \mathbf{H}(x_1, x_2, x_3) &= \mathbf{R}(x_1, x_2, x_3) + \mathbf{R}(x_1 + x_2, x_3) + \\ &\quad \mathbf{R}(x_1 + x_3, x_2) + \mathbf{R}(x_2 + x_3, x_1) + \mathbf{R}(x_1 + x_2 + x_3). \end{aligned}$$

Finally, set

$$\mathbf{G}(x_1, \dots, x_n) = \sum_{S \subset \{1, \dots, n\}} \mathbf{H}(x_i)_{i \in S} \mathbf{H}(x_i)_{i \notin S}, \quad (2.6)$$

where the summation is over all subsets S and $\mathbf{H}(x_i)_{i \in S}$ denotes the function \mathbf{H} in variables x_i , $i \in S$. For example

$$\mathbf{G}(x_1, x_2) = 2\mathbf{H}(x_1, x_2) + 2\mathbf{H}(x_1)\mathbf{H}(x_2).$$

The function map_g has a natural extension to disconnected surfaces Σ by multiplicativity. We have the following formula, see the last formula in Section 2.1.8 of [12],

Proposition 2.2. *We have*

$$G(x_1, \dots, x_n) = \sum_{\Sigma} \text{map}_{\Sigma}(2x_1, \dots, 2x_n), \quad (2.7)$$

where the summation is over all orientable surfaces Σ , including disconnected ones.

For example,

$$G(x_1, x_2) = \sum_g \text{map}_g(2x_1, 2x_2) + \sum_{g_1, g_2} \text{map}_{g_1}(2x_1) \text{map}_{g_2}(2x_2).$$

2.3

Definition 2.3. Let $Q(x_1, \dots, x_n)$, $n = 1, 2, \dots$, be a sequence of function which are symmetric in their arguments. We define

$$\begin{aligned} Q^{\circ}(x_1) &= Q(x_1), \\ Q^{\circ}(x_1, x_2) &= Q(x_1, x_2) - Q(x_1)Q(x_2), \\ Q^{\circ}(x_1, x_2, x_3) &= Q(x_1, x_2, x_3) - Q(x_1)Q(x_2, x_3) - Q(x_2)Q(x_1, x_3) \\ &\quad - Q(x_3)Q(x_1, x_2) + 2Q(x_1)Q(x_2)Q(x_3), \end{aligned}$$

and so on, namely, terms of degree d in Q come with the coefficient $(-1)^{d-1}(d-1)!$ which is the well known Möbius function for the partially ordered set Π_n of partitions of $\{1, \dots, n\}$. We call these functions Q° the *connected* part of Q .

It is clear that the connected part of the sum (2.7) is the following

$$G^{\circ}(x_1, \dots, x_n) = \sum_g \text{map}_g(2x_1, \dots, 2x_n).$$

It is clear from (2.3) that we have

Proposition 2.4.

$$\mathcal{F}(x_1, \dots, x_n) = \frac{2^{n/2-1} \pi^{n/2}}{\prod x_i^{1/2}} G^{\circ}\left(\frac{x}{2^{1/3}}\right).$$

Now it only remains to check that

$$\mathcal{G} \stackrel{?}{=} \frac{1}{2} G^{\circ}. \quad (2.8)$$

2.4

The sum \mathbf{G} can be interpreted in a diagrammatic way as follows. We take our variables $\{x_i\}$ and divide them first in two large groups in all possible ways as in (2.6). Next we divide each group into smaller subgroups, which are the blocks of the partition α in (2.5). These blocks are then grouped together into the cycles of permutations which appear in the determinant in (2.4).

The contribution of each such diagram to \mathbf{G} is a product over the connected pieces of the diagram. Therefore, in the function \mathbf{G}° every disconnected diagram will cancel out and \mathbf{G}° will be the sum over connected diagrams only.

The corresponding summands are the following. First, in (2.6) every nontrivial subset S leads to a disconnected diagram and so the connected part of \mathbf{G} is just twice the connected part of \mathbf{H} . After that, in (2.5) we can take any partition α , but then in (2.4) we need our permutation to have only one cycle.

Denote the contribution of one long cycle to (2.4) by

$$\mathbf{E}(x_1, \dots, x_n) = \int_{\mathbb{R}^n} e^{(x,z)} \prod_{i=1}^n \mathbf{K}(z_i, z_{i+1}) dz, \quad (2.9)$$

with the understanding that $x_{n+1} = x_1$. Note the invariance of the function (2.9) under the cyclic permutation of variables.

The connected part of \mathbf{R} is therefore

$$\mathbf{R}^\circ(x) = (-1)^{n+1} \sum_{\sigma \in S(n)/(12\dots n)} \mathbf{E}(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

where the summation is over representatives of the right cosets of the symmetric group modulo the cyclic group generated by the permutation $(12\dots n)$. With this notation, we have the following

Proposition 2.5.

$$\mathbf{G}^\circ(x_1, \dots, x_n) = 2 \sum_{\alpha \in \Pi_n} \mathbf{R}^\circ(x_\alpha).$$

For example,

$$\mathbf{G}^\circ(x_1, x_2) = 2\mathbf{R}^\circ(x_1, x_2) + 2\mathbf{R}^\circ(x_1 + x_2) = 2\mathbf{E}(x_1 + x_2) - 2\mathbf{E}(x_1, x_2).$$

Similarly

$$\begin{aligned} \mathbf{G}^\circ(x_1, x_2, x_3) &= 2\mathbf{E}(x_1 + x_2 + x_3) - 2\mathbf{E}(x_1 + x_2, x_3) - 2\mathbf{E}(x_1 + x_3, x_2) \\ &\quad - 2\mathbf{E}(x_2 + x_3, x_1) + 2\mathbf{E}(x_1, x_2, x_3) + 2\mathbf{E}(x_1, x_3, x_2). \end{aligned}$$

2.5

Now it only remains to prove that

$$\mathcal{E} \stackrel{?}{=} \mathbf{E}.$$

Consider the function (2.9) more closely. We have the following formula, see formula (4.5) in [16],

$$\mathbf{K}(z, w) = \int_0^\infty \text{Ai}(z + a) \text{Ai}(w + a) da. \quad (2.10)$$

Substituting this into (2.9) and interchanging the order of integration we obtain the integrals of the form considered in the following

Lemma 2.6.

$$\int_{-\infty}^\infty e^{xz} \text{Ai}(z + a) \text{Ai}(z + b) dz = \frac{1}{2\sqrt{\pi x}} \exp\left(\frac{x^3}{12} - \frac{a+b}{2}x - \frac{(a-b)^2}{4x}\right)$$

Proof. Denote by f the function $f = \text{Ai}(z + a) \text{Ai}(z + b)$. The Airy function is a solution of the Airy differential equation $\text{Ai}''(z) = z \text{Ai}(z)$. It follows that

$$f_{zzzz} - (4z + 2a + 2b)f_{zz} - 6f_z + (a - b)^2 f = 0.$$

This translates into a first order ODE for $g = \int e^{xz} f dz$.

$$g_x = \left(\frac{x^2}{4} - \frac{a+b}{2} - \frac{1}{2x} + \frac{(a-b)^2}{4x^2}\right) g,$$

from which it follows that

$$g = \frac{h(a, b)}{\sqrt{x}} \exp\left(\frac{x^3}{12} - \frac{a+b}{2}x - \frac{(a-b)^2}{4x}\right),$$

where $h(a, b)$ is some function of a and b . The obvious equations

$$f_{aa} - (z + a)f = 0, \quad f_{bb} - (z + b)f = 0$$

translate into the equations

$$g_{aa} - ag = g_{bb} - bg = g_x,$$

which imply that h is constant. The fact that $h = \frac{1}{2\sqrt{\pi}}$ follows by taking the Laplace transform of (2.10) with $z = w$ and comparing it with the known formula for it, see Section 2.6 of [12]. \square

This lemma concludes the proof of the theorem.

2.6 Examples and remarks

2.6.1

In the simplest example $n = 1$, there is no quadratic term in the exponential and we obtain the formula

$$\mathcal{E}(x) = \frac{1}{2\sqrt{\pi}} \frac{\exp\left(\frac{1}{12}x^3\right)}{\sqrt{x}} \int_0^\infty e^{xy} dy = \frac{1}{2\sqrt{\pi}} \frac{\exp\left(\frac{1}{12}x^3\right)}{x^{3/2}},$$

which corresponds to the well-known result

$$\langle \tau_{3g-2} \rangle = \frac{1}{24^g g!}, \quad \mathcal{F}(x) = \frac{\exp(x^3/24)}{x^2}.$$

2.6.2

The case $n = 2$ reduces to the ordinary error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du,$$

as follows. The following integral

$$\begin{aligned} \iint_0^\infty \exp(-P(a+b) - Q(a-b)^2) da db = \\ \frac{\sqrt{\pi}}{2P\sqrt{Q}} \exp\left(\frac{P^2}{4Q}\right) \left(1 - \operatorname{erf}\left(\frac{P}{2\sqrt{Q}}\right)\right) \end{aligned} \quad (2.11)$$

can be computed by introducing the variables $u = a + b$ and $v = a - b$ and integrating first in u and then in v .

The integral in (1.3) for $n = 2$ becomes the integral (2.11) with the following choice of parameters

$$P = \frac{x_1 + x_2}{2}, \quad Q = \frac{x_1 + x_2}{4x_1x_2}.$$

Therefore

$$\mathcal{E}(x_1, x_2) = \frac{1}{2\sqrt{\pi}} \frac{\exp((x_1 + x_2)^3/12)}{(x_1 + x_2)^{3/2}} \left(1 - \operatorname{erf} \left(\frac{1}{2} \sqrt{x_1x_2(x_1 + x_2)} \right) \right)$$

It follows that

$$\begin{aligned} \mathcal{G}(x_1, x_2) &= \mathcal{E}(x_1 + x_2) - \mathcal{E}(x_1, x_2) \\ &= \frac{1}{2\sqrt{\pi}} \frac{\exp((x_1 + x_2)^3/12)}{(x_1 + x_2)^{3/2}} \operatorname{erf} \left(\frac{1}{2} \sqrt{x_1x_2(x_1 + x_2)} \right). \end{aligned} \quad (2.12)$$

Now we use the expansion

$$e^{x^2/4} \operatorname{erf}(x/2) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} x^{2k+1},$$

which can be proved, for example, by noting that the function $\operatorname{erf}(x/2)$ satisfies the equation $u_{xx} + \frac{x}{2} u_x = 0$ and hence the function $e^{x^2/4} \operatorname{erf}(x/2)$ satisfies the equation $u_{xx} - \frac{x}{2} u_x - \frac{1}{2} u = 0$. It follows that

$$\mathcal{F}(x_1, x_2) = \frac{1}{x_1 + x_2} \exp \left(\frac{x_1^3}{24} + \frac{x_2^3}{24} \right) \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} \left(\frac{1}{2} x_1x_2(x_1 + x_2) \right)^k. \quad (2.13)$$

By the string equation

$$\mathcal{F}(x_1, x_2) = \frac{\mathcal{F}(x_1, x_2, 0)}{x_1 + x_2},$$

and so again we find agreement with the known formula for $\mathcal{F}(x_1, x_2, 0)$, see [6].

2.6.3

Observe that the only non-polynomial term in (2.13), which is $\frac{1}{x_1 + x_2}$ comes from the exceptional case $(g, n) = (0, 2)$.

Also observe that the term $\mathcal{E}(x_1 + x_2)$ cancels out completely in (2.12) with a part of \mathcal{F} . This cancellation can be seen a-priori as follows: all coefficient of \mathcal{F} are rational numbers and the term $\mathcal{E}(x_1 + x_2)$ has the wrong power of π . This argument works in general to identify terms making no contribution to \mathcal{G} .

In fact, the formula (1.4) is full of cancellations of this type. Several criteria for identifying irrelevant terms will be developed in the next section where we will be dealing with the KdV equations.

3 KdV hierarchy

3.1 Strategy

3.1.1

In this section we will give a derivation of the Witten's KdV equations which uses our current machinery and the Fay identities techniques developed by Adler, Shiota, and van Moerbeke in [1]. Namely, we will prove that the series expansion of $\mathcal{F}(x)$ about $x = 0$ is given by coefficients of one specific KdV τ -function. See [11], and also for example [4], for the exposition of how this was established originally using Kontsevich's matrix model.

3.1.2

We will need some qualitative facts about the series expansion of $\mathcal{G}(x)$ about $x = 0$, the first one being that all exponents in that expansion are half-integers. Consequently, after sufficiently many differentiations, every terms in this expansion blows up as $x_i \rightarrow +0$ for any $i = 1, \dots, n$. In other words, the coefficients of $\mathcal{G}(x)$ can be determined from singularities of \mathcal{G} and its derivatives as $x_i \rightarrow +0$.

3.1.3

Proposition 2.5 and (2.8) express the function \mathcal{G} in terms of the function \mathcal{R} which, by definition, is the following Laplace transform

$$\mathcal{R}(\xi_1, \dots, \xi_n) = \int_{\mathbb{R}^n} e^{(\xi, x)} \det [\mathcal{K}(x_i, x_j)] dz. \quad (3.1)$$

Hence the $x \rightarrow 0$ singularities of \mathcal{G} and its derivatives are determined by the similar singularities of \mathcal{R} . Clearly, the singularities of \mathcal{R} and its derivatives as $\xi_i \rightarrow +0$, are determined by the $x \rightarrow -\infty$ asymptotics of the integrand in (3.1).

The $x \rightarrow -\infty$ asymptotics of the integrand in (3.1) can be computed from the classical asymptotics of the Airy function, however it is rather complicated and difficult to use in (3.1) directly. Fortunately, there are several qualitative criteria, a sort of selection rules, which help identify many terms in the asymptotics of $\det [\mathcal{K}(x_i, x_j)]$ as irrelevant.

3.1.4

Observe that because all exponents in the expansion of $\mathcal{G}(x)$ are half-integers, no information is lost by computing $\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) \mathcal{G}(x)$ instead of \mathcal{G} . Indeed, the operator

$$\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \in \text{End}_{\mathbb{C}} \left(\sqrt{x_1 \cdots x_n} \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \right)$$

has no kernel. On the other side of the Laplace transform (3.1), this means that we are free to multiply the integrand by any factor of the form $(x_i - x_j)$.

This is convenient for the following reason. The expression of \mathcal{G} in terms of \mathcal{R} involves terms which are Laplace transforms of distributions supported on diagonals like the $\mathcal{E}(x_1 + x_2)$ term in (2.12). Also, the asymptotics of the integrand in (3.1) contains denominators of the form $\frac{1}{x_i - x_j}$ which require special handling resulting in certain contributions to the asymptotics from the diagonals. Since these contributions disappear after multiplying by appropriate factors of the form $(x_i - x_j)$, they always balance out in the final answer like they did in (2.12).

In other words, this argument shows that if a term in the asymptotics of $\det [\mathcal{K}(x_i, x_j)]$ becomes negligible after multiplication by $(x_i - x_j)$, then it can be discarded.

We will call such negligible terms the *diagonal* terms. For example, Theorem 1 can be restated as saying that

$$\mathcal{F}(x_1, \dots, x_n) = \frac{(-1)^{n+1}(2\pi)^{n/2}}{\prod x_i^{1/2}} \mathcal{E}^\circ\left(\frac{x}{2^{1/3}}\right) + \text{diagonal terms}.$$

Another qualitative way to eliminate the diagonal contributions is to observe that they come with the wrong power of π , see Section 2.6.3.

3.1.5

Another category of negligible terms are the *oscillating* terms in the asymptotics of $\det[\mathbf{K}(x_i, x_j)]$. The $x \rightarrow -\infty$ asymptotics of the Airy function $\text{Ai}(x)$ is oscillating and most of the terms in the asymptotics of the integrand in (3.1) will have factors like $\exp\left(\frac{4}{3}ix_k^{3/2}\right)$ for some $k = 1, \dots, n$. We now observe that such a term will never blow up as $\xi_k \rightarrow +0$, even if we multiply it by an arbitrary large power of x_k . Indeed, we can assume that we already got rid of all denominators as explained in 3.1.4 above, and then it is enough to show that

$$\int_c^\infty e^{-\xi x + ix^{3/2}} x^a dx = O(1), \quad \xi \rightarrow +0,$$

where $c > 0$ and a is arbitrary. After a change of variables, it becomes equivalent to

$$\int_c^\infty e^{-\xi x^{2/3}} e^{ix} x^a dx = O(1), \quad \xi \rightarrow +0,$$

for some other constants a and c . Now, if we integrate by parts integrating e^{ix} and differentiating the rest, we can decrease a so that to make the integral with $\xi = 0$ absolutely converging, thus proving our assertion.

3.1.6

Finally, we are interested in terms in $\mathcal{G}(x)$ of positive degree in all x_i . Recall that all terms of $\mathcal{G}(x)$ have positive degree in all x_i 's with the exception of the $g = 0$ terms for $n = 1, 2$.

3.2 Asymptotics of the Airy kernel

3.2.1

Introduce the following functions

$$a(x) = \sum_{k=0}^{\infty} a_k x^{-3k}, \quad \dot{a}(x) = \sum_{k=0}^{\infty} \dot{a}_k x^{-3k+1},$$

where $a_0 = \dot{a}_0 = 1$ and

$$a_k = (-i)^k \frac{(6k-1)!!}{72^k (2k)!}, \quad \dot{a}_k = \frac{1+6k}{1-6k} a_k.$$

Also set, by definition,

$$A(x) = \exp(i \frac{2}{3} x^3) a(x), \quad \dot{A}(x) = \exp(i \frac{2}{3} x^3) \dot{a}(x).$$

and

$$\mathbf{a}(x, y) = \frac{a(x) \dot{a}(y) - a(y) \dot{a}(x)}{x - y}, \quad \mathbf{A}(x, y) = \frac{A(x) \dot{A}(y) - A(y) \dot{A}(x)}{x - y},$$

We have (see e.g. [7]) the following $x \rightarrow +\infty$ asymptotics

$$\begin{aligned} \text{Ai}(-x) &\sim \frac{1}{2\sqrt{\pi} i x^{1/4}} \left[e^{\pi i/4} A(x^{1/2}) - e^{-\pi i/4} A(-x^{1/2}) \right], \\ \text{Ai}'(-x) &\sim -\frac{1}{2\sqrt{\pi} x^{1/4}} \left[e^{\pi i/4} \dot{A}(x^{1/2}) - e^{-\pi i/4} \dot{A}(-x^{1/2}) \right]. \end{aligned}$$

This asymptotics remains valid for complex x such that $|\arg x| < \frac{2}{3}\pi$.

3.2.2

It follows that away from the diagonal $x = y$ we have

$$\mathbf{K}(-x, -y) \sim \frac{1}{4\pi i x^{1/4} y^{1/4}} \sum_{\varepsilon, \varepsilon' = \pm 1} i^{(\varepsilon + \varepsilon')/2} \frac{\mathbf{A}(\varepsilon \sqrt{x}, \varepsilon' \sqrt{y})}{\varepsilon \sqrt{x} + \varepsilon' \sqrt{y}},$$

where the sum is over 4 possible combinations of the signs ε and ε' . Because the function $\mathbf{K}(-x, -y)$ is analytic at $x = y$ and its asymptotics remains valid in the complex domain, this asymptotics can be extended, using contour integrals, to the diagonal $x = y$.

By multilinearity of the determinant, we have

$$\det [\mathbf{K}(-x_i, -y_j)]_{1 \leq i, j \leq n} \sim \frac{1}{(4\pi i)^n \prod_i x_i^{1/4} y_i^{1/4}} \sum_{\varepsilon, \varepsilon' \in \{\pm 1\}^n} \det \left[\frac{\mathbf{A}(\varepsilon_i \sqrt{x_i}, \varepsilon'_j \sqrt{y_j})}{\varepsilon_i \sqrt{x_i} + \varepsilon'_j \sqrt{y_j}} \right], \quad (3.2)$$

where the summation is over 4^n choices of signs vectors ε and ε' . Observe that by definition of \mathbf{A} we have

$$\det \left[\frac{\mathbf{A}(x_i, y_j)}{x_i + y_j} \right] = \exp \left(\frac{2}{3} i \sum (x_i^3 + y_i^3) \right) \det \left[\frac{\mathbf{a}(x_i, y_j)}{x_i + y_j} \right].$$

This last determinant can be simplified using the higher Fay identities [1] which will be discussed in the following section.

3.3 Fay identities and τ -functions

3.3.1

In this section we collect, for the reader's convenience, some background material about τ -functions and Fay identities [1]. We will slightly deviate from the notational conventions of the book [9] by V. Kac. For our purposes, it will be convenient to use the notations of the Appendix to [13].

3.3.2

Given a matrix $M \in GL(\infty)$, we will denote by $\langle M \rangle$ the charge 0 vacuum matrix element

$$\langle M \rangle = (M v_\emptyset, v_\emptyset)$$

for the action of $GL(\infty)$ in the infinite wedge space. For any M , the function

$$\tau_M(t) = \langle \Gamma_+(t) M \rangle,$$

of the variables $t = (t_1, t_2, \dots)$, is a τ -function for the KP hierarchy. In our case, we will additionally assume that M is upper triangular, which implies that $M^* v_\emptyset = v_\emptyset$ and hence

$$\tau_M(0) = 1.$$

3.3.3

Consider the following matrix element

$$\Psi = \left\langle \prod_{i=1}^n \psi(z_i) \psi^*(w_i) M \right\rangle ,$$

where $\psi(z)$ and $\psi^*(w)$ are the standard generating functions for the fermionic operators. On the one hand we have (see Corollary 14.10 in [9] reproduced in (A.14) in [13])

$$\psi(z) \psi^*(w) = \frac{\sqrt{zw}}{z-w} \Gamma_-(\{z\} - \{w\}) \Gamma_+(\{w^{-1}\} - \{z^{-1}\}) ,$$

where

$$\{z\} = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots \right) .$$

From this and the commutation rule

$$\Gamma_+(t) \Gamma_-(s) = e^{\sum k t_k s_k} \Gamma_-(s) \Gamma_+(t)$$

it follows that

$$\Psi = \Pi_0 \frac{\Delta(z) \Delta(-w)}{\prod (z_i - w_j)} \tau \left(\sum \{w_i^{-1}\} - \sum \{z_i^{-1}\} \right) , \quad (3.3)$$

where Δ denotes the Vandermonde determinant and $\Pi_0 = \prod_i \sqrt{z_i w_i}$.

3.3.4

On the other hand, consider the operators $\psi_M(z) = M^{-1} \psi(z) M$, and similarly $\psi_M^*(w)$. Using the commutation rules and $M^* v_\emptyset = v_\emptyset$, we compute

$$\begin{aligned} \Psi &= \langle \psi_M(z_1) \cdots \psi_M(z_n) \psi_M^*(w_n) \cdots \psi_M^*(w_1) \rangle = \\ &\det \left[\langle \psi_M(z_i) \psi_M^*(w_j) \rangle \right] = \Pi_0 \det \left[\frac{\tau \left(\{w_i^{-1}\} - \{z_j^{-1}\} \right)}{z_i - w_j} \right] . \end{aligned} \quad (3.4)$$

Here the second equality is based on Wick's theorem, or equivalently, follows from the observation that, acting on the vacuum, the operators $\psi_M^*(w_j)$

remove vectors \underline{k} , where $k \in \{-\frac{1}{2}, -\frac{3}{2}, \dots\}$, and then the operators $\psi_M(z_i)$ have to put all these removed vectors back, in all possible orders.

Combining (3.3) with (3.4) and interchanging the roles of z and w , we obtain the identity

$$\det \left[\frac{\tau(\{z_i^{-1}\} - \{w_j^{-1}\})}{z_i - w_j} \right] = \frac{\Delta(z)\Delta(-w)}{\prod(z_i - w_j)} \tau(\sum\{z_i^{-1}\} - \sum\{w_i^{-1}\}) , \quad (3.5)$$

which is the form of the higher Fay identities [1] that we will need here.

3.3.5

From now on, we will be interested in one particular τ -function, see e.g. [10], corresponding to the matrix M_A which acts as follows

$$M_A \underline{k} = \begin{cases} \sum a_i \underline{k + 3i}, & k + 1/2 \text{ is even}, \\ \sum \dot{a}_i \underline{k + 3i}, & k + 1/2 \text{ is odd}. \end{cases}$$

That is, the matrix M looks like this:

$$M_A = \begin{bmatrix} \ddots & & & & & & \\ & a_0 & & a_1 & & \dot{a}_2 & \\ & & \dot{a}_0 & & \dot{a}_1 & & a_2 \\ & & & a_0 & & a_1 & \\ & & & & \dot{a}_0 & & \dot{a}_1 \\ & & & & & a_0 & a_1 \\ & & & & & & \dot{a}_0 \\ & & & & & & & \ddots \end{bmatrix}$$

where the empty spaces represent zeros. This matrix commutes with the bosonic operators α_k for k even and hence the τ -function

$$\tau_A = \tau_{M_A}$$

does not depend on t_k with k even, meaning that it is a τ -function for the KdV hierarchy. It also follows that

$$\tau_A(t - \{w\}) = \tau_A(t + \{-w\}) \quad (3.6)$$

for any w .

3.3.6

It follows from definitions that

$$\tau_A(\{x^{-1}\} + \{y^{-1}\}) = \sum_{i,j=0}^{\infty} a_i \dot{a}_j \frac{x^{-3i-1}y^{-3j} - y^{-3i-1}x^{-3j}}{x^{-1} - y^{-1}} = -\mathbf{a}(x, y),$$

which by (3.6) and the Fay identity (3.5) implies that

$$\det \left[\frac{\mathbf{a}(x_i, y_j)}{x_i + y_j} \right] = (-1)^n \frac{\Delta(x) \Delta(y)}{\prod (x_i + y_j)} \tau_A \left(\sum \{x_i^{-1}\} + \sum \{y_i^{-1}\} \right). \quad (3.7)$$

3.4 Asymptotics and KdV equations

3.4.1

Combining the formulas (3.2) and (3.7), we obtain the following asymptotics

$$\det [\mathbf{K}(-x_i^2, -y_j^2)]_{1 \leq i,j \leq n} \sim \frac{1}{(-4\pi i)^n \prod_i \sqrt{x_i y_i}} \sum_{\varepsilon, \varepsilon' \in \{\pm 1\}^n} e^{\frac{2}{3}i \sum (\varepsilon_i x_i^3 + \varepsilon'_i y_i^3)} \times \frac{\Delta(\varepsilon x) \Delta(\varepsilon' y)}{\prod (\varepsilon_i x_i + \varepsilon'_j y_j)} \tau_A \left(\sum \left\{ \frac{\varepsilon_i}{x_i} \right\} + \sum \left\{ \frac{\varepsilon'_i}{y_i} \right\} \right), \quad (3.8)$$

where εx denotes the term-wise product of two vectors.

Since the left-hand side in (3.8) is analytic and (3.8) is valid in the complex domain, it can be extended to the asymptotics of the integrand in (3.1) by letting $y_i \rightarrow x_i$ in (3.8).

The full form of the $y_i \rightarrow x_i$ limit of (3.8) is rather messy, but fortunately, we can concentrate only on a small fraction of the terms that survive all selection rules discussed in Section 3.1.

3.4.2

Most importantly, we get nonoscillating terms only if $\varepsilon' = -\varepsilon$ which cuts the number of summands from 4^n down to 2^n . These 2^n terms can be conveniently organized using the following operators ∇_i .

Let $f(x_i, y_i, \dots)$ be a function of x_i, y_i and some other variables which is supersymmetric in x_i and y_i in the sense that its restriction to the diagonal

$$\delta_i f = f|_{y_i=x_i}$$

does not depend on x_i , that is,

$$\frac{\partial}{\partial x_i} \delta_i f = 0. \quad (3.9)$$

Introduce the following operator

$$[\nabla_1 f](x_1, \dots) = \lim_{y_1 \rightarrow x_1} \frac{1}{2} \frac{f(x_1, y_1, \dots) - f(-x_1, -y_1, \dots)}{x_1 - y_1}.$$

Note that the numerator here vanishes on the diagonal $y_1 = x_1$, so this limit is well defined.

With this notation, we see that

$$\det [\mathbf{K}(-x_i^2, -x_j^2)]_{1 \leq i, j \leq n} \sim \frac{1}{(-2\pi i)^n \prod_i x_i} \nabla_1 \nabla_2 \cdots \nabla_n \Phi + \dots \quad (3.10)$$

where

$$\Phi = \prod_i \phi_i \prod_{i \neq j} \phi_{ij} \tau_A \left(\sum \left\{ \frac{1}{x_i} \right\} - \sum \left\{ \frac{1}{y_i} \right\} \right), \quad (3.11)$$

$$\phi_i = \exp\left(\frac{2}{3}i(x_i^3 - y_i^3)\right), \quad \phi_{ij} = \frac{(x_i - x_j)(y_j - y_i)}{(x_i - y_j)(x_j - y_i)}, \quad (3.12)$$

and dots in (3.10) stand for oscillating terms. Observe that all factors in (3.11), (3.12) have the supersymmetry (3.9).

3.4.3

The operators ∇_i commute and satisfy the following Leibnitz-like rule

$$\nabla_i(f \cdot g) = \nabla_i(f) \cdot \delta_i(g) + \delta_i(f) \cdot \nabla_i(g). \quad (3.13)$$

Let us examine the effect of applying ∇_i to (3.11).

Denote by τ_μ the coefficients in the expansion of the τ_A

$$\tau_A = \sum_{\mu} \frac{\tau_{\mu} t_{\mu}}{|\text{Aut } \mu|} \quad (3.14)$$

where the summation is over all partitions μ , $|\text{Aut } \mu|$ is the product of factorials of multiplicities of parts in μ , $t_{\mu} = \prod t_{\mu_i}$. The variables t_k are specialized in (3.11) in the following way

$$t_k = \frac{1}{k} \sum x_i^{-k} - \frac{1}{k} \sum y_i^{-k}. \quad (3.15)$$

Recall that τ_A depends only on t_k with k odd. It is clear that

$$\nabla_i t_k = -x_i^{-k-1}, \quad k \text{ odd}, \quad (3.16)$$

and also that δ_i simply removes x_i and y_i from the sum (3.15).

Similarly, it is clear that

$$\nabla_i \phi_i = 2ix_i^2, \quad (3.17)$$

Finally, for the last type of factors in (3.11) we have

$$\delta_i \phi_{ij} = \delta_j \phi_{ij} = 1$$

which, in particular implies that

$$\nabla_i \delta_j \phi_{ij} = \nabla_j \delta_i \phi_{ij} = 0.$$

It follows that the ∇_i 's have to be applied to ϕ_{ij} 's in pairs to get a nonzero result

$$\nabla_i \nabla_j \phi_{ij} = \frac{x_i^2 + x_j^2}{(x_i^2 - x_j^2)^2}. \quad (3.18)$$

3.4.4

The factor corresponding to (3.18) in the asymptotics of $\det [\mathbf{K}(-x_i, -x_j)]$ will be

$$\frac{1}{\sqrt{x_i x_j}} \frac{x_i + x_j}{(x_i - x_j)^2}, \quad (3.19)$$

where the first term comes from the prefactor in (3.10). This has a second order pole on $x_i = x_j$, which in the full asymptotics of (3.10) cancels out with poles of oscillating terms. This singularity of (3.19) is immaterial and can be removed as explained in Section 3.1.4. What is important about (3.19) is that it has degree -2 in x_i and x_j and, hence, whatever contribution it makes to the asymptotics of (3.1), it will be in degree 0 in ξ_i and ξ_j . Since we are interested in terms of strictly positive degree in all variables, see Section 3.1.6, the terms containing (3.19) are negligible. By the same token, any terms containing (3.17) can be also discarded.

It follows that relevant terms in the asymptotics (3.10) are obtained by applying all operators ∇_i to the τ -function. From (3.16) and the Leibnitz rule (3.13) we obtain

$$\nabla_1 \cdots \nabla_n t_\mu = \begin{cases} (-1)^n |\text{Aut } \mu| m_\mu, & \ell(\mu) = n, \\ 0 & \text{otherwise,} \end{cases}$$

where m_μ denotes the monomial symmetric function

$$m_\mu = \frac{1}{|\text{Aut } \mu|} \sum_{\sigma \in \mathcal{S}(n)} \prod x_{\sigma(i)}^{\mu_i}.$$

Therefore, modulo irrelevant terms, the asymptotics of the integrand in (3.1) is

$$\det [\mathbf{K}(-x_i, -x_j)] \sim \frac{1}{(-2\pi i)^n \prod x_i} \sum_{\ell(\mu)=n} \tau_\mu m_\mu(x^{-1/2}) + \dots$$

Recall that only partitions μ with odd parts enter this sum.

3.4.5

Now it remains to use the formulas

$$\int_c^\infty e^{-\xi x} x^{a-1} dx \sim \frac{\Gamma(a)}{\xi^a}, \quad \xi \rightarrow +0,$$

and

$$\Gamma\left(-\frac{2k+1}{2}\right) = \frac{(-2)^{k+1} \sqrt{\pi}}{(2k+1)!!}$$

to obtain the expansion

$$\mathbf{R}(\xi) = \frac{\prod \xi^{1/2}}{i^n \pi^{n/2}} \sum_{m_1, \dots, m_n} \tau_{2m+1} \prod \frac{(-2\xi_i)^{m_i}}{(2m_i+1)!!} + \dots \quad (3.20)$$

where dots stand for irrelevant terms, that is, for diagonal terms and terms of negative degree in the ξ_i 's, and τ_{2m+1} is the coefficient in (3.14) corresponding to the nonincreasing rearrangement of the numbers $2m_i + 1$.

By (2.8) and Proposition 2.5 we have

$$\mathcal{G}(x) = \sum_{\alpha \in \Pi_n} \mathbf{R}^\circ(x_\alpha) = \mathbf{R}^\circ(x) + \dots,$$

where R° is the connected part of R and dots stand for diagonal terms. Now (3.20) and Theorem 1 imply that

$$\mathcal{F}(x) = \left(\sum_{m_1, \dots, m_n} \tilde{\tau}_{2m+1} \prod \frac{x_i^{m_i}}{(2m_i + 1)!!} \right)^\circ + \dots,$$

where $\tilde{\tau}$ denotes the rescaled τ -function

$$\tilde{\tau}_\mu = (-i)^{|\mu|} 2^{|\mu|/3} \tau_\mu, \quad (3.21)$$

circle stands for the connected part, and dots stand for terms of negative degree in x , that is, for the $g = 0$ terms in the case $n = 1, 2$. This concludes the proof of the KdV equations.

Note that the rescaling (3.21) is equivalent to the following rescaling

$$a_k \mapsto \frac{(6k - 1)!!}{36^k (2k)!},$$

keeping the relation $\dot{a}_k = \frac{1+6k}{1-6k} a_k$.

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